DCDS supplement, 2007, pp. 122–129
Proceedings 6th AIMS Internat. Conference (Poitiers, France)
http://aimsciences.org/article/doi/10.3934/proc.2007.2007.122

ASYMPTOTIC BEHAVIOR OF TIME OPTIMAL ORBITAL TRANSFER FOR LOW THRUST 2-BODY CONTROL SYSTEM

Alex Bombrun and Jean-Baptiste Pomet

INRIA Sophia
2004 route des lucioles
BP 93, 06902, Sophia Antipolis, France

Abstract. In the studies of low thrust time optimal orbital transfer it was conjectured that, when the thrust modulus tends to zero, the product of this modulus by the minimum transfer time admits a finite limit. The purpose of the present note is to better frame the nature of this asymptotic behavior and to prove this conjecture.

1. **Introduction.** We study orbital transfer for satellites, whose model (2-body control system) is a mass point subject to the gravitation field of a central body and an extraneous force, the thrust, as a control; in practice, this thrust is produced by some sort of engine and its magnitude is limited by the technology. In the case of chemical engines, the thrust modulus is much larger than the gravitational force and control laws can be designed in term of impulsive controls. For the new generation of rockets like plasmic engines, this approximation is not relevant because the maximum magnitude of the thrust –let us call it ε– is much smaller than the gravitational force; new control laws should be designed.

Given a maximal thrust modulus ε, the time optimal transfer problem between two elliptic orbits is of primary interest. The optimal synthesis is far from being solved, and quite an ambitious objective; let us however mention the efficient algorithms to compute numerical solutions described in [12] and [7], and a deep geometrical analysis proposed in [6]. Call T_ε the value of the minimum transfer time (the two elliptic orbits are fixed); based on numerical experiments and heuristic reasoning, it was conjectured in [8] that $\varepsilon T_\varepsilon$ tends to a finite limit when $\varepsilon \to 0$. The present paper is devoted to proving this conjecture. The main tool is the construction of an average control system, via a new averaging procedure that we introduced in [3]. In a forthcoming paper, we will write a detailed study of this average control system.

The averaging method is a powerful tool to understand some dynamical system with fast and slow dynamics [2]. Among the numerous publications on averaging adapted to control theory let us mention the work of François Chaplais [10], where

2000 *Mathematics Subject Classification.* Primary: 49M99, 93C10; Secondary: 70F99.

Key words and phrases. 2-body control system, Low thrust, Time optimal problem, Orbital transfer, Averaging.

This work was partially supported, and motivated, by *Alcatel Alenia Space, Cannes, France*, under a contractual research project.

it is shown that the solutions of the optimal problems

$$\begin{cases} \dot{x} = f(x, u, t, t/\varepsilon)\,, \quad u \in U \subset \mathbb{R}^m \\ \min \int_0^T \phi(x, u) dt \quad , \end{cases} \tag{1}$$

with fast variation are close in norm L^∞ to the solutions of the averaged system

$$\begin{cases} \dot{x} = \frac{1}{w} \int_0^w f(x, u(t, \theta), t, \theta) d\theta \\ \min \int_0^T \left(\frac{1}{w} \int_0^w \phi(x, u(t, \theta)) d\theta \right) dt \end{cases} \tag{2}$$

under some assumptions, in particular $U = \mathbb{R}^m$ (relaxed constraint on the control), local optimality of extremals and a coercive cost. Although these assumptions are not satisfied for the time optimal orbital transfer problem (see (11)), Richard Epenoy and Sophie Geffroy used the method from [10] to develop an algorithm and a software (Mipelec) to compute numerical approximation of time optimal orbital transfers [11]. To our knowledge, proofs of convergence (when low thrust goes to zero) are not available, but this was a significant advance in practically computing time optimal trajectories for this system.

Following the same line for the so-called energy problem (minimizing the L^2 norm of the control between two non-degenerate elliptic orbits), Bernard Bonnard and Jean-Baptiste Caillau noted in [6] that the averaged dynamics is the one of the geodesic flow of a Riemannian metric, and more remarkably that this flow is completely integrable; they gave explicit expressions at least for planar transfers.

We use another approach in order to prove the theorem stated in the section 2.3. Instead of studying the averaged equation of the extremals for the time optimal problem, we build an "average control system" that approximates *all* the solutions of the original systems with low thrust.

The paper is organized as follows: Section 2 presents the time optimal transfer problem for the 2-body control system and states the main result, we then construct in Section 3 an average control system (also useful in more general situations) and use this construction to prove the main result in Section 4.

2. **The 2-body control system.** Let $\mathbb{R}^3_* = \mathbb{R}^3 \setminus \{0\}$. The Newton equations read:

$$\begin{cases} \dot{\mathbf{r}} &= \mathbf{v}\,, \\ \dot{\mathbf{v}} &= -\mu \frac{\mathbf{r}}{\|\mathbf{r}\|^3} + \mathbf{u}\,, \end{cases} \qquad \|\mathbf{u}\| \leq \varepsilon\,, \tag{3}$$

where $(\mathbf{r}, \mathbf{v}) \in \mathbb{R}^3_* \times \mathbb{R}^3$ is the satellite position-velocity vector, the control u is the (small) acceleration produced by the low thrust engine and $\varepsilon > 0$ parameters the problem. Decomposing u on an orthonormal frame $\xi\eta\zeta$ in the acceleration space: $u = u_\xi \xi + u_\eta \eta + u_\zeta \zeta$, (3) can be rewritten more compactly as:

$$\frac{dq}{dt} = f_0(q) + u_\xi f_\xi(q) + u_\eta f_\eta(q) + u_\zeta f_\zeta(q), \; \sqrt{u_\xi^2 + u_\eta^2 + u_\zeta^2} \leq \varepsilon\,, \tag{4}$$

where $q \in T\mathbb{R}^3_*$ ($(\mathbf{r}, \mathbf{v})$ are particular coordinates for q), the "*drift*" vector field f_0 is the gravitational vector field responsible of the Keplerian motion ($\dot{\mathbf{r}} = \mathbf{v}$, $\dot{\mathbf{v}} = -\mu\mathbf{r}/\|\mathbf{r}\|^3$ in the coordinates of (3)) and f_ξ, f_η and f_ζ are vector fields on $T\mathbb{R}^3_*$.

2.1. **First integrals.** Let $H = \|\mathbf{v}\|^2/2 - \mu/\|\mathbf{r}\|$ denote the mechanical energy, $\mathbf{c} = \mathbf{r} \times \mathbf{v}$ the kinetic momentum and

$$\mathcal{C}_- = \{q \in T\mathbb{R}^3_*,\, H < 0, \mathbf{c} \neq 0\} \quad \text{and} \quad \mathbf{e} = \frac{\mathbf{v} \times \mathbf{c}}{\mu} - \frac{\mathbf{r}}{\|\mathbf{r}\|} \tag{5}$$

the "elliptic domain" and the eccentricity vector respectively. The free motion $\dot{q} = f_0(q)$ is well understood (see [14] or many textbooks); in particular all solutions with initial condition $q(0) = q_0 \in \mathcal{C}_-$ are defined for all times and periodic with closed trajectories in $T\mathbb{R}^3_*$ projecting onto ellipses in $\mathbb{R}^3_*$. The scalar H and the vectors $\mathbf{c}$, $\mathbf{e}$ provide seven first integrals that are not independent; many choices of five independent integrals are possible; for the sake of completeness, let us describe a possible choice of such first integrals, to be used as privileged coordinates for the control system (4). Let $\mathbf{XYZ}$ be an orthonormal frame, (c_x, c_y, c_z) the coordinate of $\mathbf{c}$ in this frame, $c = \|\mathbf{c}\|$ and R the rotation around the vector $\mathbf{c} \times \mathbf{Z}$ that maps $\mathbf{c}$ to the direction $\mathbf{Z}$; set $(h_x, h_y) = (-\frac{c_y}{c+c_z}, \frac{c_x}{c+c_z})$ and, since $R\,\mathbf{e}$ belongs to the plane $\mathbf{XY}$, e_x, e_y may be defined by $R\,\mathbf{e} = e_x\mathbf{X} + e_y\mathbf{Y}$. Adding the cumulated longitude L (see [9, 16] for a precise definition), we obtain a coordinate chart $(c, e_x, e_y, h_x, h_y, L)$ valid on the sub-domain of the non degenerated elliptic domain $\mathcal{C}_- \setminus \{c_z = -c\}$. This domain is parametrized by $(c, e_x, e_y, h_x, h_y, L) \in \mathbb{R}^+_* \times \mathbb{D} \times \mathbb{R}^2 \times \mathbb{R}$, with $\mathbb{D} = \{x \in \mathbb{R}^2, \|x\| < 1\}$. Using for $\xi\eta\zeta$ the orthoradial frame QSW associated to the satellite (Q is the radial direction, S the orthoradial direction and W the normal direction), we have the following analytic expression of the vector fields in (4) :

$$f_0 = \frac{\mu^2 Z^2}{c^3}\frac{\partial}{\partial L}, \tag{6}$$

$$f_S = \frac{c^2}{\mu Z}\frac{\partial}{\partial c} + \frac{cA}{\mu Z}\frac{\partial}{\partial e_x} + \frac{cB}{\mu Z}\frac{\partial}{\partial e_y}, \tag{7}$$

$$f_Q = \frac{c}{\mu}\sin L\frac{\partial}{\partial e_x} - \frac{c}{\mu}\cos L\frac{\partial}{\partial e_y}, \tag{8}$$

$$f_W = \frac{cY}{\mu Z}\left(-e_y\frac{\partial}{\partial e_x} + e_x\frac{\partial}{\partial e_y} + \frac{\partial}{\partial L}\right) + \frac{cX}{2\mu Z}\left(\cos L\frac{\partial}{\partial h_x} + \sin L\frac{\partial}{\partial h_y}\right) \tag{9}$$

where
$$\begin{aligned} A &= e_x + (1+Z)\cos L\,, & Z &= 1 + e_x\cos L + e_y\sin L\,,\\ B &= e_y + (1+Z)\sin L\,, & X &= 1 + h_x^2 + h_y^2\,,\\ & & Y &= h_x\sin L - h_y\cos L\,. \end{aligned}$$

The interest of using these coordinates is that, by rectifying the vector field f_0, it emphasizes the slow and fast dynamics, where slow comes from ε being small. We can group the slow variables in $I = (c, e_x, e_y, h_x, h_y)$, while the angle L is fast. Note that we could have used other choices of five independent first integrals for I. We are soon going to re-write (4) in the condensed matrix form (12).

2.2. **Controllability.** This system is controllable. Indeed it satisfies the strong accessibility condition:

$$\forall q \in T\mathbb{R}^3_*,\ \mathrm{Rank}_q\{f_\xi, f_\eta, f_\zeta, [f_0, f_\xi], [f_0, f_\eta], [f_0, f_\zeta]\} = 6\,. \tag{10}$$

This condition implies in particular that the linear approximation along a Keplerian orbit is controllable. Moreover the drift, being periodic, is recurrent in the elliptic domain $\mathcal{C}_-$. Hence we can apply the theorem for affine systems with recurrent drift to conclude of the controllability in the elliptic domain (Theorem 5 of the chapter 4 in [13]). In [4] we proved that the controllability in elliptic domain implies the controllability in the full phase plane.

2.3. **Time optimal problem.** Let $\mathbb{K}$ be a compact set that is the topological closure of a connected open set in $\mathcal{C}_-$. The transfer problem in optimal time restricted to $\mathbb{K}$ between two elliptic orbits, $I_0, I_1 \in \mathbb{K}$, is defined by the following set of equations

$$
\begin{aligned}
&\frac{dq}{dt} = f_0(q) + u_Q f_Q(q) + u_S f_S(q) + u_W f_W(q)\,, \quad \sqrt{u_Q^2 + u_S^2 + u_W^2} \le \varepsilon, \\
&\min T\,, \quad q(0) \in \{I_0\} \times \mathbb{S}^1,\ q(T) \in \{I_1\} \times \mathbb{S}^1, \quad \forall t,\ q(t) \in \mathbb{K} \times \mathbb{S}^1\,.
\end{aligned}
\tag{11}
$$

The Filippov theorem (see Theorem 10.1 page 138 in [1]) implies the existence of a time optimal transfer. Note that the existence result requires the restriction to a compact set of the non-degenerated elliptic domain. Otherwise there is no warranty that optimal trajectories are not going through the origin $\mathbf{r} = 0$. Let T^ε the solution of (11) we will prove the following theorem conjectured in [8],

Main Theorem. *For a fixed set $\mathbb{K}$, the product $\varepsilon\, T^\varepsilon$ is converging toward a finite positive limit when ε tends to* 0.

This is a particular case of Theorem 2, stated for the more general systems described in section 2.4. See the paragraph immediately after (13).

2.4. **Keplerian control systems.** Let n, m be two integers and consider the control system with state $(I, \theta) \in \mathcal{O} \times \mathbb{R}$, with $\mathcal{O}$ an open set in $\mathbb{R}^n$, and control $u = (u_1, \ldots, u_m)$ in $\mathbb{R}^m$:

$$
\begin{aligned}
\dot{I} &= F(I, \theta)\, u \\
\dot{\theta} &= w(I, \theta) + g(I, \theta)\, u\,, u \in \mathbb{R}^m, u \in B_1^m,
\end{aligned}
\tag{12}
$$

with $B_1^m = \{u \in \mathbb{R}^m, \|u\| \le 1\}$, $F(I, \theta)\, u = \sum_{i=1}^m u_i f_i(I, \theta)$ and $g(I, \theta)\, u = \sum_{i=1}^m u_i g_i(I, \theta)$ (obvious matrix notations) where, for each i, $\hat{f}_i : (I, \theta) \mapsto (f_i, g_i)$ is a smooth vector fields on $\mathcal{O} \times \mathbb{R}$ and 2π-periodic relatively to θ. We assume that the pulsation w is smooth and positive. More precisely for any compact set $\mathbb{K} \subset \mathcal{O}$, there is an $\alpha > 0$ such that, for all $I \in \mathbb{K}$ and $\theta \in \mathbb{R}$, $w(I, \theta) > \alpha > 0$. Moreover we assume that

$$
\forall \theta \in [0, 2\pi], \forall I \in \mathcal{O}, \mathrm{Rank} \left\{ \frac{\partial^j F}{\partial \theta^j}(I, \theta),\ j \in \mathbb{N} \right\} \ = \ n. \tag{13}
$$

We call these systems *Keplerian control systems.* The control system (4) can be written in this form with $n = 5$, $m = 3$, $\theta = L$, $I = (c, e_x, e_y, h_x, h_y)$ and F, g, w defined by (6)-(9) for I in $\mathbb{R}_*^+ \times \mathbb{D} \times \mathbb{R}^2$. Moreover the strong accessibility condition (10) is equivalent to the rank condition (13). Hence, the following controllability result holds.

Proposition 1. *Let $\varepsilon > 0$, for all I_1 and I_2 in $\mathcal{O}$ there is a finite time T and an admissible control $u \in L^1_{\mathrm{loc}}([0, T], B_1^m)$ that joins them.*

We now perform the following rescaling on system (12) :

$$
\theta = \frac{\phi}{\varepsilon}\,, \quad u = \varepsilon\, v\,, \quad t = \frac{\tau}{\varepsilon}\,, \tag{14}
$$

to obtain ($'$ stands for $d/d\tau$) :

$$
\begin{aligned}
I' &= F(I, \phi/\varepsilon)\, v \\
\phi' &= w(I, \phi/\varepsilon) + \varepsilon\, g(I, \phi/\varepsilon)\, v\,, \|v\| \le 1.
\end{aligned}
\tag{15}
$$

Since, for ε small enough, ϕ' remains positive, we may invert τ and ϕ and obtain the following normal form for (12):

$$\frac{dI}{d\phi} = \frac{F}{w+\varepsilon\, g\, v}\, v\,, \tag{16}$$

$$\frac{d\tau}{d\phi} = \frac{1}{w+\varepsilon\, g\, v}\,, \qquad \|v\| \le 1\,. \tag{17}$$

3. **Average control system.** The 2-body control system evolves like the Keplerian motion when the control is zero. Hence when the control is small the motion should be close to Keplerian orbits. In this section we define an average control system and state a comparison theorem between the trajectories of the original and the average systems. All the statements are given without proofs, detailed proofs and developments can be found in [3].

3.1. **Definition and convergence.** By setting

$$\mathcal{F}(I,\theta) = \frac{F(I,\theta)}{w(I,\theta)} \text{ and } \mathcal{R}(I,\theta,\varepsilon,v) = -\frac{g(I,\theta)\, v\, F(I,\theta)}{(w(I,\theta)+\varepsilon\, g(I,\theta)\, v)\; w(I,\theta)}\,, \tag{18}$$

we can write (16) as a family of control systems indexed by ε defined on $\mathbb{R}^n$:

$$\frac{dI}{ds} = \mathcal{F}(I, s/\varepsilon)\, u + \varepsilon\, \mathcal{R}(I,s,\varepsilon,u), \quad u \in B_1^m\,. \tag{19}$$

The average control system associated to the family (19) is, by definition, the differential inclusion

$$\frac{dI}{ds} \in \mathcal{E}(I)\,, \tag{20}$$

where, for each $I \in \mathbb{R}^n$, the set of all admissible velocities is

$$\mathcal{E}(I) = \left\{ \frac{1}{2\pi}\int_0^{2\pi} \mathcal{F}(I,\theta)\, u(\theta)\mathrm{d}\theta,\ u(.) \in L^1_{\mathrm{loc}}(\mathbb{S}^1, B_1^m) \right\}\,. \tag{21}$$

A solution of (20) is an absolutely continuous map $s \mapsto I(s)$ such that the inclusion (20) holds for almost all s. Note that along a solution, the control function u can be chosen measurable with respect to s and θ jointly.

Now we can state a convergence result of trajectories of the family of control systems (19) when ε tends to zero.

Theorem 1. *For some $I_0 \in \mathbb{R}^n$, let $\mathbb{K}$ be a compact set in $\mathbb{R}^n$ containing I_0 such that for all ε, all solutions I of (19) initialized at $I(0) = I_0$ stay in the interior of $\mathbb{K}$ for all $t \in [0,T]$.*

1. *Let $t \to I^0(t)$ a solution of (19) defined on $[0,T]$ initialized at $I^0(0) = I_0$. There exists a family of controls $u_\varepsilon(.) \in L^1([0,T], B_1^m)$ such that the family of solutions $I^\varepsilon(t)$ of (19) with $u = u_\varepsilon(t)$ and $I^\varepsilon(0) = I(0)$ converges uniformly on $[0,T]$ to $I^0(.)$ when ε tends to 0.*
2. *Conversely, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of real positive numbers converging to 0 and a sequence of controls $u_n(.) \in L^1([0,T], B_1^m)$. Let $I^n(.)$ the sequence of solutions of (19) with $\varepsilon = \varepsilon_n$, $u = u_n(t)$ and $I^n(0) = I_0$. If $I^n(.)$ is uniformly converging on $[0,T]$ toward $I^\star(.)$, then $I^\star$ is a solution of (20).*

Note that this theorem can also be set in terms of accessible sets, the set of all point accessible at a time τ from a given points. Let $\mathcal{A}^{\epsilon}_{I_0,\tau}$ denote the accessible set for original systems (19) and $\mathcal{A}^{0}_{I_0,\tau}$ the accessible set of the average system (20), then there exists $k > 0$ such that, for all $\epsilon > 0$

$$d(\mathcal{A}^{\epsilon}_{I_0,\tau}, \mathcal{A}^{0}_{I_0,\tau}) \quad \leq \quad k\,\epsilon \tag{22}$$

where d denotes the Hausdorff distance between two compact sets.

3.2. **Properties of average control systems.**

Proposition 2. *The set $\mathcal{E}(I)$ is compact, convex, symmetric with respect to the origin. Let $I \in \mathbb{R}^n$, under the hypothesis* (13) *the interior of $\mathcal{E}(I)$ is non empty.*

The condition (13) is equivalent to the controllability of the linear approximation of the original systems along the constant trajectory I obtained with the zero control.

Proposition 3. *If the condition* (13) *is satisfied in all points of $\mathbb{R}^n$ then the system* (20) *is globally controllable.*

4. **Proof of the main result.** In this section, we prove the main theorem (Section 2.3) by stating Theorem 2, which is more general and is a consequence of Lemmae 1 and 2.

4.1. **Minimum time.** Let $\mathbb{K}$ be a compact that is the topological closure of a connected open set in $\mathcal{O}$, and let I_1, I_0 be two orbits in $\mathbb{K}$.

Consider the time optimal transfer problem restricted to the compact $\mathbb{K}$ for the family of systems (12) that we write in a compact form

$$(12)_{\varepsilon}\ , \quad I(0) = I_0,\, I(T) = I_1,\, I([0,T]) \subset \mathbb{K}, \quad \min T\,, \tag{23}$$

This problem is equivalent to the problem (11) when you consider the particular case of the 2-body control system.

Then, consider the time optimal transfer problem for the average system

$$(20)\ , \quad I(0) = I_0,\, I(\phi_1) = I_1,\, I([0,\phi_1]) \subset \mathbb{K}, \quad \min \tau(\phi_1)\,, \tag{24}$$

where the “time” τ is defined by

$$\frac{d\tau}{d\phi} \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\theta}{w(I(\phi),\theta)}\ , \quad \tau(0) = 0\,, \tag{25}$$

with $\phi \mapsto I(\phi)$ a solution of the average system (20).

Theorem 2. 1. *There exists a unique $\tau_0 > 0$ solution of the problem* (24).

2. *For some $\varepsilon_0 > 0$ and for all ε, $0 < \varepsilon < \varepsilon_0$, there exists a unique T_ε solution of the problem* (23).
3. *Moreover $\varepsilon\, T_\varepsilon$ converges toward τ_0 when ε tends to* 0.

Proof. Filippov theorem implies –recall (Proposition 2) that $\mathcal{E}(I)$ is convex– the existence of the solutions to these optimal problems. The convergence is a consequence of Lemma 1. □

Consider the time optimal transfer problems restricted to $\mathbb{K}$ for the original systems in normal form

$$(16)_{\varepsilon}\ , \quad I(0) = I_0,\, I(\phi_1) = I_1,\, I([0,\phi_1]) \subset \mathbb{K}, \quad \min \tau(\phi_1)\,. \tag{26}$$

Lemma 1. *For some $\varepsilon_0 > 0$ and for all ε, $0 < \varepsilon < \varepsilon_0$, there exists an unique τ_ε solution of the problem* (26)*. Moreover τ_ε converges to τ_0 as ε tends to* 0.

Proof. The existence of τ_ε also follows from the Filippov theorem. Let $(\tilde{I}, \tilde{\tau})$ be an optimal trajectory, solution of the average problem (24) defined on $[0, \tilde{\phi}_1]$, $\tau(\tilde{\phi}_1) = \tau_0$.

The point 2 of Theorem 1 implies that there exists a sequence of trajectories $\tilde{I}_\varepsilon$ solutions of original systems (16) defined on $[0, \tilde{\phi}_\varepsilon]$ such that

$$\|\tilde{I}_\varepsilon - \tilde{I}\|_\infty \leq k\varepsilon\,. \tag{27}$$

To this sequence we associate through the equation (17) the sequence of times $\tilde{\tau}_\varepsilon$, functions of ϕ defined on $[0, \tilde{\phi}_\varepsilon]$. This sequence is uniformly converging and in particular the sequence of final times $\tilde{\tau}_\varepsilon(\tilde{\phi}_\varepsilon)$ converges. The Lemma 2 implies that for small ε each trajectory of the sequence can be prolongated into a trajectory that joins I_1 in a time lower than $\tilde{\tau}_\varepsilon(\tilde{\phi}_\varepsilon) + c_2\varepsilon$. Let $(\tilde{I}_\varepsilon, \tilde{\tau}_\varepsilon)$ be the sequence of these prolongated trajectories. The sequence $\tilde{\tau}_\varepsilon(\tilde{\phi}_\varepsilon)$ converges toward τ_0 when ε tends to zero, moreover by optimality hypothesis

$$\tilde{\tau}_\varepsilon(\tilde{\phi}_\varepsilon) \geq \tau_\varepsilon\,, \tag{28}$$

hence the sequence of optimal times τ_ε is uniformly bounded.

Conversely, let $(\overline{I}_\varepsilon, \overline{\tau}_\varepsilon)$ be a sequence of trajectories, solutions of the originals problems (26). The sequence of derivatives $d\overline{I}_\varepsilon/d\phi$ is uniformly bounded weakly relative to ε, hence the sequence of solutions is equicontinuous and the Ascoli theorem implies that there exists a uniformly converging sub-sequence. Let $(\overline{I}_\varepsilon, \overline{\tau}_\varepsilon)$ denote this subsequence defined on $[0, \overline{\phi}_\varepsilon]$ and let $(\overline{I}, \overline{\tau})$ be the limit trajectory defined on $[0, \overline{\phi}_1]$. According to the point 1 of the Theorem 1, $\overline{I}$ is an admissible trajectory of the average system (20) such that $\overline{I}(0) = I_0$ and $\overline{I}(\overline{\phi}_1) = I_1$; hence

$$\overline{\tau}(\overline{\phi}) \geq \tau_0\,. \tag{29}$$

As the sequences $\tilde{\tau}_\varepsilon(\tilde{\phi}_\varepsilon)$ and $\overline{\tau}_\varepsilon(\overline{\phi}_\varepsilon)$ converge respectively toward τ_0 and $\overline{\tau}$, the inequalities (29) and (28) imply $\overline{\tau} = \tau_0$. □

4.2. **In the neighborhood of the target orbit.** The hypothesis (13) implies that the optimal balls are locally equivalent to euclidean balls. Let $c_1 > 0$ and $I_1 \in \mathbb{K}$, we define the balls,

$$\mathcal{B}_\varepsilon^{c_1} = \{I, \|I - I_1\| \leq c_1\varepsilon\}. \tag{30}$$

and the mapping $T^\varepsilon : \mathbb{R}^n \to \mathbb{R}$ that associates to an initial orbit $I \in \mathbb{K}$ the minimum transfer time to I_1 for the originals systems (16).

Lemma 2. *There exists $\varepsilon_0 > 0$ and $c_2 > 0$ for all $\varepsilon \leq \varepsilon_0$, such that*

$$\sup_{I \in \mathcal{B}_\varepsilon^{c_1}} T^\varepsilon(I) \leq c_2\varepsilon \tag{31}$$

Proof. The rank condition (13), equivalent to (10), implies controllability of the linearized system of along the constant trajectory $I(t) = I_1$.

Let $Pf : L^\infty([0,1], U) \to \mathbb{R}^n$ be the end point mapping that associates to a control u the point $I(1)$ of the trajectory $I(t)$ solution of (16) initialized at $I(0) = I_1$.

Since the linearized system is controllable, this mapping is a smooth submersion at $u = 0$, continuously differentiable with a full rank derivative according to the theorem 1 page 57 in [15]. The rank theorem 52 page 464 in [15] implies the existence of a "right inverse" function continuously differentiable, i.e. there exists a neighborhood $\mathcal{V}_{I_1}$ of I_1 and a continuously differentiable mapping $\sigma : \mathcal{V}_{I_1} \to L^\infty([0,1], U)$ such that $\sigma(I_1) = 0$ and $Pf(\sigma(I)) = I$ for all points $I \in \mathcal{V}_{I_1}$.

The constant c_2 can be chosen equal to the Lipschitz constant of the mapping Pf^{-1} in a compact neighborhood $\tilde{\mathcal{V}}_{I_1}$ of I_1 included in $\mathcal{V}_{I_1}$ and the constant ε_0 chosen such that $\mathcal{B}^{c_1}_{\epsilon_0}$ is a subset of $\tilde{\mathcal{V}}^c_{I_1}$. □

REFERENCES

[1] A. A. Agrachev and Y. L. Sachkov, *Control theory from the geometric viewpoint*, volume 87 of *Encyclopaedia of Mathematical Sciences*, Springer-Verlag, Berlin, 2004, Control Theory and Optimization, II.

[2] V. I. Arnold, *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, second edition, 1989, Translated from the Russian by K. Vogtmann and A. Weinstein.

[3] A. Bombrun, *Transferts orbitaux en poussée faible, optimalité et stabilisation*, Ph.D. thesis, Mines de Paris, 2007.

[4] A. Bombrun, J.-B. Pomet and M. Sigalotti, *Mechanical systems and rendez-vous controllability*, Proceedings of 44th IEEE Conference on Decision and Control-ECC, 2005.

[5] B. Bonnard and J.-B. Caillau, *Riemannian metric of the averaged energy minimization problem in orbital transfer with low thrust*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **24**(3) (2007), 395–411.

[6] B. Bonnard, J.-B. Caillau and E. Trélat, *Geometric optimal control of elliptic keplerian orbits*, Discrete and continous dynamical systems-series B, **5**(4) (2005), 929–956.

[7] J.-B. Caillau, *Contribution à l'étude du contrôle en temps minimal des transferts orbitaux*, PhD thesis, Institut national polytechnique de Toulouse, 2000.

[8] J.-B. Caillau, J. Gergaud and J. Noailles, *Minimum time control of the kepler equation*, Unsolved Problems in Mathematical Systems and Control Theory, V. D. Blondel and A. Megretski, editors, Princeton University Press, 2004.

[9] J.-P. Carrou, *Mécanique Spaciale*, volume 1, CNES, 1995.

[10] F. Chaplais, *Averaging and deterministic optimal control*, SIAM J. Control And Optimization, (1987).

[11] S. Geffroy, *Généralisation des techniques de moyennation en contrôle optimal- Application aux problèmes de transfert et rendez-vous orbitaux à poussée faible*, Thèse de doctorat, Institut National Polytechnique de Toulouse, 1997.

[12] S. Geffroy and R. Epenoy, *Optimal low-thrust transfers with constraints–generalization of averaging techniques*, Acta Astronautica, **41**(3) (1997), 133–149.

[13] V. Jurdjevic, *Geometric Control Theory*, volume 51 of Cambridge Studies in Advanced Mathematics, Cambridge Univ. Press, 1997.

[14] J. Milnor, *On the geometry of the kepler problem*, American Mathematical Monthly, **90** (1983), 353–365.

[15] E. D. Sontag, *Mathematical control Theory: Deterministic Finite Dimensional Systems*, Springer, 1998.

[16] O. Zarrouati, *Trajectoires spaciales*, CNES, 1987.

Received September 2006; revised February 2007.

E-mail address: Jean-Baptiste.Pomet@sophia.inria.fr